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Published in:
IEEE Transactions on Automatic Control

DOI:
[10.1109/TAC.2012.2208314](https://doi.org/10.1109/TAC.2012.2208314)

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Document Version
Publisher's PDF, also known as Version of record

Publication date:
2013

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Citation for published version (APA):

Cao, M., Zhang, S., & Camlibel, M. K. (2013). A Class of Uncontrollable Diffusively Coupled Multiagent Systems with Multichain Topologies. *IEEE Transactions on Automatic Control*, 58(2), 465-469.
<https://doi.org/10.1109/TAC.2012.2208314>

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A Class of Uncontrollable Diffusively Coupled Multiagent Systems with Multichain Topologies

Ming Cao, Shuo Zhang, and M. Kanat Camlibel

Abstract—We construct systematically a class of uncontrollable diffusively coupled multiagent systems with a single leader and multichain topologies. For studying the controllability of diffusively coupled multiagent systems, such identified uncontrollable systems serve as counterexamples that prove the need to modify the existing sufficient condition using graph partitioning characterization. The uncontrollability of the constructed multichain structures can be preserved when the structures are further augmented to get better connected. The paper also provides an algorithm to obtain the minimal leader-invariant relaxed equitable partition for the graph associated with any diffusively coupled multiagent system guided by a single leader.

Index Terms—Controllability, diffusive coupling, equitable partition, multiagent system.

I. INTRODUCTION

Recently the model of multiagent systems have been widely used in the study of distributed and cooperative control of complex networks [1]–[3]. It is of particular interest to study the case when the agents interact with one another following some rules using only local information while desired global behaviors of the network emerge as a result [4]–[7]. While various system synthesis results have been constructed to explain collective behaviors in natural, social or engineered systems using multiagent models, researchers are also interested in controlling

the behavior of the overall system by just adjusting a small fraction of the agents in the complex system [8].

To introduce control input into such complex networks, it is natural to assign the role of "leaders" to some of the agents. Here, by *leaders* we mean those agents who are aware of a predefined common goal for the whole group and are able to influence some other agents. So one natural question to ask is whether the group-level coordination can indeed be achieved through the leaders by influencing directly or indirectly the rest of the agents. Hence, the controllability issue arises [9]–[11]. To see the importance of understanding the controllability of a multiagent system, consider the following example [2] of controlling a team of mobile robots for which, one wants to know whether it is possible to move all the robots from any initial positions to any desired final positions within finite time through manipulating the trajectories of the leaders. Similar examples exist widely in the study of various real networks as reported in [12].

When the agents' dynamics are determined by linear nearest-neighbor couplings and thus the overall system can be modeled as a linear system, it is clear from the classical systems theory that such controllability conditions can be delineated explicitly using algebraic conditions by computing the eigenstructure of the system matrix [2], [13]. However, the graph theoretical conditions relying purely on the topologies of networks are more desirable for designing distributed control strategies [2]. This underscores the importance, for the purpose of designing leader-follower type control strategies for multiagent systems, of identifying the classes of graphs that are not controllable. Encouraging progress has been made in the past few years. For example, it has been shown that star graphs with the leader being the hub and leader symmetric graphs are not controllable [11]. More recently, a graph partitioning operation is defined for the topologies of multiagent systems with leaders, and it has been indicated that the controllability properties of a given system are closely related to the leader-invariant equitable partitions [14] for the graph describing the agents' neighbor relationships.

It is an interesting observation that all the reported single-leader uncontrollable graphs in the literature share the common feature that their minimal leader-invariant equitable partitions (MLEPs) have cells of size greater than one, i.e., some cells contain more than one vertex [14]. Following the tradition in graph theory, we say a graph partition is *discrete* if each of its cells contains only one vertex. Then one may wonder whether the set of uncontrollable graphs can be characterized as those graphs whose MLEPs are not discrete. This is equivalent to ask whether a graph must be controllable if its MLEP is discrete. The main contribution of this paper is that, in contrary to some existing results [14] (for which we have written a comment note [15]), we provide a *negative* answer to this question. To be more precise, it is stated in [14] that a graph's MLEP being discrete is both necessary and sufficient for the associated multiagent system to be controllable. In [15] we have provided a concrete six-agent system as a counterexample showing that the condition is only necessary but not sufficient. Motivated by the counterexample in [15], in this paper, we are able to construct a class of uncontrollable graphs in the form of interconnected multiple chains, whose MLEPs are discrete. Such classes of graphs can be characterized and built systematically and their sizes can be large. The main technique that we exploit is to construct for the graphs, structured eigenvectors of the system matrices using only local information. The implication of the finding of such classes of graphs is twofold. First, the complete graph theoretical characterization of uncontrollable graphs is still elusive, not as indicated in [14]. Second, when designing topologies of networks, some systems can be identified to be uncontrollable without computing the spectrums of their system matrices. In

Manuscript received January 05, 2011; revised July 13, 2011; accepted June 11, 2012. Date of publication July 12, 2012; date of current version January 19, 2013. This work was supported in part by grants from the Dutch Organization for Scientific Research (NWO) and by the Dutch Technology Foundation (STW). Recommended by Associate Editor H. S. Chang.

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Digital Object Identifier 10.1109/TAC.2012.2208314

the process of proving the main result of the paper, we develop an algorithm to find the MLEP for a given graph, which is motivated by existing algorithms for computing the automorphisms of graphs [16]. Note that we are not aware of an existing algorithm that is readily capable to obtain the MLEP of a graph.

The rest of the paper is organized as follows. We first review the model for single-leader diffusively coupled networks in Section II. We then provide discussion on chain graphs and multichain graphs in Section III. An algorithm to find the MLEP is presented in Section IV, where the algorithm is applied to the multichain graphs. It is then shown that the MLEP of a system's graph being discrete is only necessary but not sufficient for the system to be controllable. In addition, we generalize our discussion to graphs that contain multichain graphs as subgraphs in Section V.

II. SINGLE-LEADER DIFFUSIVELY COUPLED MULTIAGENT SYSTEMS AND THEIR CONTROLLABILITY

As in [11], we assign the roles of leaders and followers to the agents in a multiagent system. In particular, we consider the multiagent system consisting of $n > 0$ followers, labeled by $1, \dots, n$, and one leader, labeled by $n+1$. Let $x_i \in \mathbb{R}$, $i \in \{1, \dots, n+1\}$, denote the state of agent i . For a pair of distinct agents i and j , $1 \leq i \leq n$ and $1 \leq j \leq n+1$, we say agent j is a neighbor of agent i if $x_j(t)$, $t \geq 0$, is known by agent i . We assume that the neighbor relationships are fixed during the evolution of the system. Since the focus of the paper is to understand how the neighbor relationships affect the system's controllability, we only consider simple agent dynamics; more specifically, the followers' dynamics are assumed to be governed by linear diffusive couplings

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i), \quad 1 \leq i \leq n \quad (1)$$

where \mathcal{N}_i is the set of indices of the neighbors of agent i , and the leader's state is determined by an exogenous control signal $u \in \mathbb{R}$ such that

$$x_{n+1} = u. \quad (2)$$

Now let $x = [x_1 \ \dots \ x_n]^T$, and then the dynamics (1)–(2) can be written into a compact form

$$\dot{x} = Ax + bu \quad (3)$$

where for $i, j \in \{1, \dots, n\}$ and $i \neq j$, the elements of A and b are defined by

$$b_i = \begin{cases} 1 & \text{if } n+1 \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ij} = \begin{cases} 1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

and $a_{ii} = -\sum_{j=1}^n a_{ij} - b_i$.

Note that for (3), the column vector b is uniquely determined by

$$b = -A\mathbf{1} \quad (4)$$

where $\mathbf{1}$ is the all-one vector.

We associate (3) with a graph \mathbb{G} with vertex set $\mathcal{V} = \{1, \dots, n+1\}$ and edge set \mathcal{E} such that there is an edge in \mathbb{G} from vertex j to i for any $i, j \in \mathcal{V}$ if and only if $j \in \mathcal{N}_i$. In view of the fact that the leader is always labeled by $n+1$ and the definitions for a_{ij} , we know that there is a one-to-one correspondence between the matrix A and its associated graph \mathbb{G} . Given a graph \mathbb{G} , we can always write down

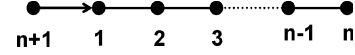


Fig. 1. Chain graph.

its associated matrix $A(\mathbb{G})$ and conversely given a matrix A , we can always draw its associated graph $\mathbb{G}(A)$. Let \mathbb{G}_F denote the subgraph of \mathbb{G} that is induced by the set $\mathcal{V}_F = \{1, \dots, n\}$ consisting of the vertices corresponding to the followers. In the sequel, we consider the case when \mathbb{G}_F is undirected.

In view of (4) and the one-to-one correspondence between A and $\mathbb{G}(A)$, the controllability of the pair (A, b) can be deduced from the topological information contained in $\mathbb{G}(A)$. In the sequel, for (3), we say the graph $\mathbb{G}(A)$ is *controllable* if and only if (A, b) is controllable. For the sake of simplicity, we call the spectrum of A the *spectrum of the graph* $\mathbb{G}(A)$.

In the next section, we present a class of uncontrollable graphs with simple chain structures.

III. CHAIN GRAPHS

We define a *chain graph* with $n+1$ vertices to be the graph for which one can label its vertices in such a way that the edge set contains exactly the edge $(n+1, 1)$ and the edges $(i, i-1)$, $(i-1, i)$, $1 < i \leq n$. We call n the *length* of the chain.

The following result reveals some relationships between the spectrums of two chain graphs if their lengths satisfy certain relationships.

Lemma 1: If λ is an eigenvalue of $A(\mathbb{G}_1)$ with associated eigenvector v , where \mathbb{G}_1 is the chain graph with $n+1$ vertices (see Fig. 1). Then λ is also an eigenvalue of $A(\mathbb{G}_2)$, where \mathbb{G}_2 is the chain graph with $k(2n+1) + n + 1$ vertices, $k = 1, 2, \dots$. In addition, one can construct from v an eigenvector \bar{v} of $A(\mathbb{G}_2)$ associated with λ .

Proof: Let $v = [v_1 \ v_2 \ \dots \ v_n]^T$ be the eigenvector associated with λ for $A(\mathbb{G}_1)$. Now we construct the $(k(2n+1) + n)$ -dimensional column vector \bar{v} as follows: For $1 \leq i \leq k(2n+1) + n$, the i th element of \bar{v} is

$$\bar{v}_i = \begin{cases} v_1 & \text{if } i \bmod 2(2n+1) = 1 \text{ or } 2n \\ v_2 & \text{if } i \bmod 2(2n+1) = 2 \text{ or } 2n-1 \\ \vdots & \vdots \\ v_n & \text{if } i \bmod 2(2n+1) = n \text{ or } n+1 \\ 0 & \text{if } i \bmod (2n+1) = 0 \\ -v_1 & \text{if } i \bmod 2(2n+1) = 2n+2 \text{ or } 4n+1 \\ -v_2 & \text{if } i \bmod 2(2n+1) = 2n+3 \text{ or } 4n \\ \vdots & \vdots \\ -v_n & \text{if } i \bmod 2(2n+1) = 3n+1 \text{ or } 3n+2 \end{cases} \quad (5)$$

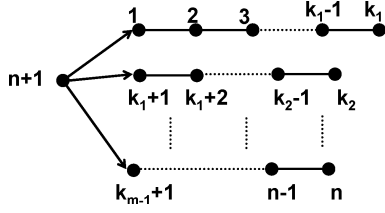
Using the fact that $A(\mathbb{G}_1)v = \lambda v$, one can show λ is an eigenvalue of $A(\mathbb{G}_2)$ with an associated eigenvector \bar{v} since

$$A(\mathbb{G}_2)\bar{v} = \lambda\bar{v}$$

as a result of the construction (5). This completes the proof. \square

We say a graph \mathbb{G} with $n+1$ vertices is an *m-chain graph* $m > 1$, if one can label its vertices in such a way that there exist integers $1 \leq k_1 < k_2 < \dots < k_{m-1} < n$ such that $\mathcal{E}(\mathbb{G})$ is the union of the edge set $\{(n+1, 1), (n+1, k_1+1), \dots, (n+1, k_{m-1}+1)\}$ and the edge set $\{(i-1, i), (i, i-1), 1 < i \leq n \text{ and } i \neq 1, k_1+1, \dots, k_{m-1}+1\}$. A typical *m-chain graph* is shown in Fig. 2.

Lemma 2: If \mathbb{G} is an *m-chain graph* and the length of each chain i , $1 \leq i \leq m$, is $3l_i + 1$ for some $l_i \geq 0$, then $A(\mathbb{G})$ has -1 as an eigenvalue whose geometric multiplicity is at least m .

Fig. 2. m -chain graph.

Proof: It suffices to construct m independent vectors $v^i \in \mathbb{R}^n$, $1 \leq i \leq m$, such that $Av^i = -v^i$. It is easy to check that for a chain graph of length one, 1 is a one-dimensional eigenvector of its eigenvalue -1 . Consider the chain with vertices $n+1$ and $1, \dots, 3l_1+1$, then from Lemma 1, one can construct $\bar{v}^1 \in \mathbb{R}^{3l_1+1}$ in the form of (5). Now let $v^1 = [(\bar{v}^1)^T \ 0 \ \dots \ 0]^T \in \mathbb{R}^n$, then it is easy to check that $Av^1 = -v^1$. Similarly, for each chain we can construct a v^i and all these v^i are orthogonal to each other. \square

Now we show that some multichain graphs are not controllable.

Lemma 3: If \mathbb{G} is an m -chain graph and the length of each chain i is $3l_i + 1$ for some $l_i \geq 0$, then \mathbb{G} is not controllable.

The proof of this lemma makes use of the following result.

Lemma 4: [2, Proposition 10.3] System (3) is uncontrollable if A has an eigenvalue whose geometric multiplicity is greater than one.

Proof of Lemma 3: The conclusion follows directly from Lemma 4 since $A(\mathbb{G})$ has an eigenvalue -1 whose geometric multiplicity is at least $m > 1$ as proven in Lemma 2. \square

Now we show some special properties of the partitions of multichain graphs.

IV. GRAPHS WITH DISCRETE MLEPS

For a graph \mathbb{G} of (3) and a constant $1 \leq r \leq n+1$, we call \mathcal{V} 's nonempty, disjoint subsets C_i , $1 \leq i \leq r$, a *partition* of \mathcal{V} if $\bigcup_i C_i = \mathcal{V}$. Accordingly, we call r the *size* of the partition and C_i *cell* i . Let π denote a partition with cells C_1, \dots, C_r . Motivated by the related definitions in [14], we say π is a *relaxed equitable partition* (REP) if for any $i \neq j \in \{1, \dots, r\}$, each vertex in C_i has the same number of neighbors in C_j . If in addition $C_r = \{n+1\}$, π is said to be a *leader-invariant REP* (LEP). Among all the LEPs of \mathbb{G} , we call the one with the smallest size a *minimal LEP* (MLEP).

Motivated by the classical algorithms for computing graph automorphisms [16], we first present an algorithm to find the MLEP for any given graph and show that the MLEP is in fact unique for any graph. Consider a graph $\mathbb{G}(\mathcal{V}, \mathcal{E})$. For two non-empty and disjoint subsets \mathcal{V}_1 and \mathcal{V}_2 of \mathcal{V} , we say \mathcal{V}_1 can be *split* with respect to \mathcal{V}_2 if \mathcal{V}_1 can be partitioned into non-empty, disjoint subsets $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_k$, $k > 1$, satisfying (a) each vertex in \mathcal{U}_i , $1 \leq i \leq k$, has the same number of neighbors in \mathcal{V}_2 and (b) any pair of vertices that are in different \mathcal{U}_i 's have different numbers of neighbors in \mathcal{V}_2 .

Now we describe the algorithm to obtain the MLEP for the graph \mathbb{G} .

Step 1: Let $\mathcal{V}_1 = \{1, \dots, n\}$ and $\mathcal{V}_0 = \{n+1\}$. Split \mathcal{V}_1 with respect to \mathcal{V}_0 . If \mathcal{V}_1 cannot be split, then set $k = 1$ and go to step 4. Otherwise, replace \mathcal{V}_1 by its subsets obtained by the split operation and go to the next step.

Step 2: Relabel the subsets in the current partition by $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_k$, $k > 1$, in such a way that (a) $\mathcal{V}_0 = \{n+1\}$ and (b) for $1 \leq i \leq k$,

$$\left\{ v = \min_{u \in \mathcal{V} - \mathcal{V}_0 \cup \mathcal{V}_1 \cup \dots \cup \mathcal{V}_{i-1}} u \right\} \subseteq \mathcal{V}_i.$$

Set $j = 1$, and go to the next step.

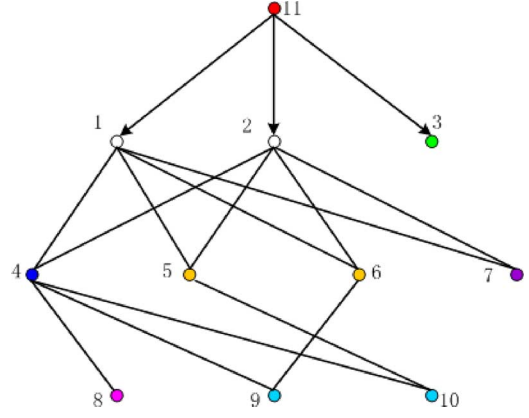


Fig. 3. MLEP obtained by the splitting algorithm for a random graph.

Step 3: For $1 \leq i \leq k$, split \mathcal{V}_i with respect to \mathcal{V}_j . If any \mathcal{V}_i can be split, replace all such \mathcal{V}_i 's by their partitioned subsets and go back to the beginning of step 2. Otherwise, set $j = j + 1$. If $j > k$, go to the next step; otherwise, return to the beginning of step 3.

Step 4: Return the current partition $\{\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_k\}$.

We call the four-step algorithm just described the *splitting algorithm*. Fig. 3 shows the MLEP of an eleven-vertex random graph obtained by the splitting algorithm where vertices in the same cell are denoted by the same color.

We now prove results related to the splitting algorithm.

Proposition 1: For a graph \mathbb{G} of (3), the partition of its vertex set \mathcal{V} obtained by carrying out the splitting algorithm is a LEP.

Proof: Let $\pi = \{\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_k\}$ denote the partition after splitting \mathcal{V} . If each \mathcal{V}_i , $1 \leq i \leq k$, contains only one element, then π is obviously a LEP. Now consider the case when there is at least one \mathcal{V}_i having more than one element. For any such fixed i , choose any pair of distinct elements $v_1, v_2 \in \mathcal{V}_i$. Then it suffices to prove that v_1 and v_2 have the same number of neighbors in any set \mathcal{V}_j , $j \neq i$. Suppose the contrary is true. Then there is a set \mathcal{V}_p , $1 \leq p \leq k$ and $p \neq i$, such that v_1 and v_2 have different numbers of neighbors in \mathcal{V}_p . Consequently, it must be true that \mathcal{V}_i can be split by \mathcal{V}_p , and hence step 3 in the splitting algorithm can still be performed. This contradicts the fact that π is obtained after carrying out the splitting algorithm and thus no split operation can be further carried out on \mathcal{V}_i . We arrive at the conclusion. \square

The result stated in the previous proposition can be further strengthened into the following result.

Proposition 2: For a graph \mathbb{G} of (3), the partition of its vertex set \mathcal{V} obtained by carrying out the splitting algorithm is an MLEP.

Let $\pi_1 = \{\mathcal{V}_1, \dots, \mathcal{V}_p\}$ and $\pi_2 = \{\mathcal{U}_1, \dots, \mathcal{U}_q\}$ be two partitions of \mathcal{V} . We say π_1 is *finer* than π_2 or equivalently π_2 is *coarser* than π_1 , denoted by $\pi_1 \preceq \pi_2$, if for any $1 \leq i \leq p$, \mathcal{V}_i is always a subset of \mathcal{U}_j for some $1 \leq j \leq q$. Then obviously, $p \geq q$ where the equality sign holds if and only if $\pi_1 = \pi_2$. We prove the following result about the relationships between the cells of any two partitions, one of which is finer than the other.

Lemma 5: Consider two partitions $\pi_1 = \{\mathcal{V}_1, \dots, \mathcal{V}_p\}$ and $\pi_2 = \{\mathcal{U}_1, \dots, \mathcal{U}_q\}$, for which $\pi_1 \preceq \pi_2$. Then each cell \mathcal{U}_i , $1 \leq i \leq q$, of π_2 can be written as the union of some cells of π_1 .

Proof: Consider any fixed $1 \leq i \leq q$. We list the elements of \mathcal{U}_i in ascending order as $\mathcal{U}_i = \{u_1, \dots, u_r\}$ for some $r \geq 1$. Then u_1 must also be an element of a cell in π_1 since π_1 is also a partition of \mathcal{V} . Denote this cell by \mathcal{V}_{j_1} . Combining with the definition of a partition being finer, we know that \mathcal{V}_{j_1} must be a subset of \mathcal{U}_i . Let u_j , $1 < j \leq r$,

be the smallest element in \mathcal{U}_i that is not in \mathcal{V}_{j_1} . If such an element does not exist, then $\mathcal{U}_i = \mathcal{V}_{j_1}$ and the conclusion holds. If on the other hand, we can find such a u_j , then we can find the cell \mathcal{V}_{j_2} that contains u_j and at the same time is a subset of \mathcal{U}_i . We continue this procedure until we have constructed a sequence of set $\mathcal{V}_{j_1}, \mathcal{V}_{j_2}, \dots, \mathcal{V}_{j_s}, 1 \leq s \leq r$, such that $\mathcal{U}_i = \bigcup_{k=1}^s \mathcal{V}_{j_k}$. \square

Proof of Proposition 2: It suffices to show that for any LEP $\pi_1 = \{\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_p\}$, we always have $\pi_1 \preceq \pi$ where π is obtained by splitting \mathcal{V} . This can be proved by looking into the pair of partitions before and after a split operation. It holds that $\pi_1 \preceq \{\{1, \dots, n\}, \{n+1\}\}$, which is the starting partition when carrying out the splitting algorithm. Since the cells of π_1 can be uniquely grouped into those having $n+1$ as a neighbor and those not having $n+1$ as a neighbor, π_1 is finer than the partition obtained after splitting $\{1, \dots, n\}$ with respect to $\{n+1\}$, i.e., the partition obtained after step 1 of the splitting algorithm. Now let $\tilde{\pi}$ be a partition obtained in the middle of the execution of the splitting algorithm and $\bar{\pi}$ be the partition obtained by one split operation on $\tilde{\pi}$ in the execution of the splitting algorithm. Suppose in this split operation \mathcal{V}_p has been split with respect to \mathcal{V}_q . Now we want to prove if $\pi_1 \preceq \tilde{\pi}$, then $\pi_1 \preceq \bar{\pi}$. Since $\pi_1 \preceq \tilde{\pi}$, in view of Lemma 5, it must be true that \mathcal{V}_q is a union of some sets in π_1 . Since π_1 is a LEP, it must be true that the other sets of π_1 can be uniquely grouped according to their numbers of neighbors in \mathcal{V}_q ; in other words, splitting \mathcal{V}_p with respect to \mathcal{V}_q leads to $\bar{\pi}$ that is still coarser than π_1 . Hence, by induction we know that every splitting operation during the execution of the algorithm leads to a new partition that is coarser than π_1 , and hence the proof is complete. \square

In the proof of Proposition 2, we have in fact also proved the following equivalent result.

Corollary 1: All LEPs of \mathbb{G} are finer than the MLEP obtained by the splitting algorithm.

Now we show that the MLEP of \mathbb{G} is in fact unique.

Theorem 1: For any graph \mathbb{G} of (3), its MLEP is unique.

Proof: Let π_1 be the MLEP of \mathbb{G} obtained by the splitting algorithm. Let π_2 be any MLEP of \mathbb{G} . Then π_2 is also a LEP and from Corollary 1 we have $\pi_2 \preceq \pi_1$. Then the size of π_2 is no less than that of π_1 where the equality sign holds only when $\pi_1 = \pi_2$. Since both π_2 and π_1 are MLEPs, they must have the same size and thus $\pi_1 = \pi_2$. \square

Now we will use the splitting algorithm to show that some multichain graphs have discrete MLEPs.

Theorem 2: If the lengths of the chains of an m -chain graph \mathbb{G} are different, then its MLEP is discrete.

Proof: From Proposition 2 we know that the partition of \mathbb{G} obtained by the splitting algorithm is an MLEP. From Theorem 1, we know \mathbb{G} 's MLEP is unique. Hence, we only need to prove that the partition obtained from the splitting algorithm is discrete. Assume that the vertices of \mathbb{G} have been labeled as in Fig. 2. In step 1 of the algorithm, we first split $\mathcal{V}_1 = \{1, \dots, n\}$ with respect to $\mathcal{V}_0 = \{n+1\}$ and after the relabeling in step 2, we obtain $\mathcal{V}_0 = \{n+1\}$, $\mathcal{V}_1 = \{1, k_1+1, \dots, k_{m-1}+1\}$ and \mathcal{V}_2 consisting of the rest of the vertices. Then we split \mathcal{V}_2 with respect to \mathcal{V}_1 , and after relabeling we have \mathcal{V}_0 and \mathcal{V}_1 as before, $\mathcal{V}_2 = \{2, k_1+2, \dots, k_{m-1}+2\}$ and \mathcal{V}_3 consisting of the rest of the vertices. Since the chains are of different lengths, there is a chain with the greatest length. Let the last vertex in this chain be k_p , then in the above split operations, $\{k_p\}$ becomes a cell of size one. Then $\{k_p - 1\}$ becomes a cell of size one after splitting the subset containing $k_p - 1$ with respect to $\{k_p\}$. Following the same argument, every vertex in this longest chain comprises a cell of size one after a certain number of splitting operations in step 3 of the splitting algorithm. Then similar analysis can be applied to the vertices of the chain with the second greatest length. After applying the same argument sequentially to all the chains in the descending order of their lengths, we

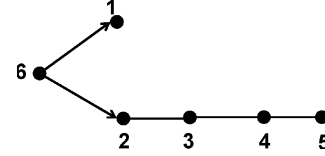


Fig. 4. An example of a two-chain graph.

know that all vertices $1, \dots, n$ are in cells of size one at the end of the execution of the splitting algorithm. \square

In view of Lemmas 3 and Theorem 2, we consider any system defined in (3) whose graph is an m -chain graph and the lengths of its chains are in the form of $3l_i + 1$ with different l_i 's. Then the MLEP of the system's graph is discrete while the system is not controllable. The following example gives such a two-chain graph with the lengths of the chains being 1 and 4 respectively. Since in [14] it has been shown that for (3) being controllable, it is necessary that the MLEP of its graph is discrete. Compared to [14] and in view of the example in Fig. 4 of an uncontrollable graph with discrete MLEP, we have gone further and proven the following result.

Theorem 3: For (3), the MLEP of its graph being discrete is only necessary but not sufficient for the system being controllable.

The implication of Theorem 3 is that for a diffusively coupled multi-agent system with a single leader, how to construct necessary and sufficient conditions for the system being controllable in terms of the system's topologies is still an open question. Nevertheless, progress has been made to provide tight bounds on the systems' controllable subspaces and for those systems with special classes of graphs called distance-regular graphs, the controllable subspaces can even be characterized precisely [17].

For any multichain graph that we have discussed so far, the follower subgraph \mathbb{G}_F is not connected. In the next section, we give several examples of *augmented* multichain graphs which have connected follower subgraphs but are still not controllable.

V. AUGMENTED MULTICHAIN GRAPHS

It turns out that some m -chain graphs can be augmented by adding edges connecting different chains. The augmentation can be carried out in such a way that the augmented graph still has a discrete MLEP and is at the same time uncontrollable. The main intuition is that by examining the eigenvectors constructed in (5), the eigenstructure will not be changed after adding edges connecting the vertices corresponding to the zero elements in the eigenvectors.

Lemma 6: Consider an m -chain graph, each chain i of which has length $3l_i + 1$ for some $l_i \geq 0$. If an edge can be added connecting some pair of vertices whose distances to vertex $n+1$ are multiples of three, then the resulted augmented m -chain graph has -1 as an eigenvalue whose geometric multiplicity is at least m .

We have shown the same property for the m -chain graphs considered in Lemma 2. In fact, Lemma 6 can be proven using the same argument as in the proof of Lemma 2 by showing that the m independent eigenvectors that we have constructed in the proof of Lemma 2 are also the eigenvectors for the augmented graph considered in Lemma 6. We provide one example of an augmented two-chain graph in Fig. 5 for which Lemma 6 is applicable.

Lemma 7: Consider an m -chain graph, each chain i of which has length $3l_i + 1$ for some $l_i \geq 0$. If an edge can be added connecting some pair of vertices whose distances to vertex $n+1$ are in the forms of $6p_1 + q_1$ and $6p_2 + q_2$ respectively for some $p_1, p_2 \geq 0, q_1 \in \{1, 2\}$, and $q_2 \in \{4, 5\}$, then the resulted augmented m -chain graph has an

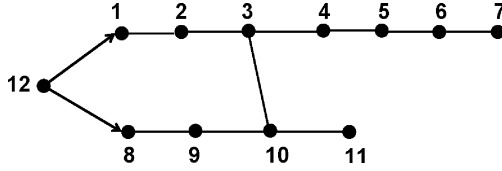


Fig. 5. An example of an augmented two-chain graph with -1 as an eigenvalue of geometric multiplicity two.

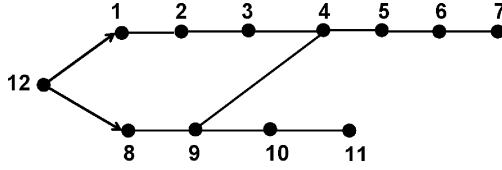


Fig. 6. Example of an augmented two-chain graph with an eigenvector that is orthogonal to b .

eigenvector associated with eigenvalue -1 that is orthogonal to the vector b defined in (3).

Proof: Let i and j with $i < j$ be the indices of the two chains that are connected by the added edge and consider the two vectors $\bar{v}^i \in \mathbb{R}^{3l_i+1}$ and $\bar{v}^j \in \mathbb{R}^{3l_j+1}$ that are in the form of (5). Then we construct the vector $v = [0 \ \dots \ 0 \ (\bar{v}^i)^T \ 0 \ \dots \ 0 \ -(\bar{v}^j)^T \ 0 \ \dots \ 0]^T$ in which \bar{v}^i and \bar{v}^j are positioned according to the indices of the vertices in chains i and j . Then one can check that v is an eigenvector associated with eigenvalue -1 and $v^T b = 0$. \square

We provide one example of an augmented two-chain graph in Fig. 6 to which Lemma 7 is applicable.

In both of the two augmentation methods considered in Lemmas 6 and 7, the graphs obtained are obviously uncontrollable. In addition, if the lengths of the chains in such an augmented multichain graphs are different, we can identify those whose MLEPs are discrete.

VI. CONCLUSION

We have constructed systematically a class of multiagent systems with multichain structures that are not controllable. Since such systems can at the same time have discrete MLEPs, we have shown that contrary to some existing result, the system's topology being discrete is only necessary but not sufficient for the system to be controllable. There are augmented multichain structures that are, though better connected, not controllable either. In the process of developing the proofs of the main results, we have also provided an algorithm to obtain the MLEP for a given system topology.

The class of uncontrollable system topologies that have been identified in this paper, is by no means exhaustive. Further work needs to be done to search for a sufficient condition involving only topological features to guarantee the controllability of the system. We will report in a related paper the progress studying systems with general linear agent dynamics instead of the restricted single-integrator dynamics studied in this paper. In that case, the system's controllability will not be determined completely by the topological features of the system's graph. Since the topic of structural controllability is closely related to the controllability and structural features of a system, we are also interested in looking into structural controllability of diffusively coupled multiagent systems.

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